Accelerating Frames of Reference and the Clock Paradox

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The Lorentz–Einstein transformations are obtained by a method which enables one to derive a coordinate transformation between an inertial frame of reference and a noninertial accelerating system. With the aid of this transformation, one computes the explicit time for the round trip of the noninertial twin who leaves the inertial twin with initial speed $-V$, and returns by means of an acceleration. It is found that the accelerating twin returns younger than the inertial twin.

I. INTRODUCTION

IN past and recent years there has been presented a number of papers dealing with Einstein's clock paradox. Some of these papers obviate the difficulties which arise in the clock paradox by referring to the general theory of relativity as regards the rate of a clock in a potential field. We propose here to find a coordinate transformation between an inertial frame of reference and an accelerating system by an analysis which is purely kinematic and is based on the invariance of the coordinate speed of light. The line element for the accelerating frame of reference will be found to be equivalent to the line element obtained in the general theory of relativity for a uniform gravitational field.

The twin problem discussed in this paper is analogous to the case in which the noninertial twin is thrown off like a ball, with an initial speed in the negative direction, and a positive acceleration during the whole time of flight, at which time the accelerating twin is again coincident with the inertial twin. The explicit proper times for the round trip journey as noted by both twins show that the accelerating twin returns younger than the inertial twin.

II. DERIVATION OF THE LORENTZ–EINSTEIN TRANSFORMATIONS

We consider an $O'XYZ$ coordinate system moving with speed V along the X axis relative to an $Oxyz$ frame of reference, O coincident with O' at $t=T=0$. The motion of a point fixed relative to O ($x=x_0=$constant) is given by $X=X_0-VT$, or $X_0=X+VT$. For each $x_0=$constant there corresponds an $X_0=$constant, and conversely. Mathematically this implies that

$$x=F(X+VT). \tag{1}$$

If dX is a measurement of length in the $O'XYZ$ system ($T=$constant), then

$$dx=F'(X+VT)dX. \tag{2}$$

The assumption that dx remain invariant if we replace V by $-V$ (isotropy of space) yields $F'(X+VT)=F'(X-VT)$ for all X, T. Of necessity, $F'(X+VT)=k=$constant, so that

$$x=k(X+VT) \tag{3}$$

with k a dimensionless constant which reduces to unity for $V=0$, and, moreover, $k(V)=k(-V)$.

Next we assume that $t=G(X,T)$, which yields

$$\frac{dx}{dt}=\frac{k[(dX/dT)+V]}{(\partial G/\partial X)(dX/dT)+(\partial G/\partial T)}. \tag{4}$$

The postulate that $dX/dT=\pm c$ implies that $dx/dt=\pm c$ (invariance of speed of light) yields

$$c^2(\partial G/\partial X)+c(\partial G/\partial T)=k(c+V)$$

and

$$c^2(\partial G/\partial X)-c(\partial G/\partial T)=k(-c+V). \tag{5}$$

It follows that

$$\partial G/\partial X=kV/c^2, \quad \partial G/\partial T=k, \tag{6}$$

which yields

$$x=k(X+VT)$$

and

$$t=k(T+VX/c^2). \tag{7}$$

The final postulate that the inverse transformation obtainable from Eq. (7) remain invariant in form by replacing V by $-V$ (group property), yields $k=(1-V^2/c^2)^{-\frac{1}{2}}$.

Thus, the Lorentz–Einstein transformations are given by

$$X=\frac{x-Vt}{(1-V^2/c^2)^{\frac{1}{2}}},\quad Y=y,\quad Z=z,\quad T=\frac{t-Vx/c^2}{(1-V^2/c^2)^{\frac{1}{2}}},\tag{8}$$

under the further assumption that meter-sticks in the directions perpendicular to the relative motion have been adjusted to read the same for both coordinate frames.

From Eq. (8) one obtains the invariant line element for inertial frames of reference, namely

$$\begin{aligned}ds^2&=c^2dt^2-dx^2-dy^2-dz^2\\&=c^2dT^2-dX^2-dY^2-dZ^2.\end{aligned}\tag{9}$$

Proper time in either coordinate system is given by $ds/c=dt$ for x,y,z constant, $ds/c=dT$ for X, Y, Z constant.

III. A COORDINATE TRANSFORMATION BETWEEN AN ACCELERATING SYSTEM AND AN INERTIAL FRAME OF REFERENCE

We consider an $O'XYZ$ frame of reference accelerating along the x axis relative to an inertial $Oxyz$ coordinate system. The two frames of reference are coincident and at rest relative to each other at $t=T=0$.

Let the motion of a point P fixed in the $Oxyz$ system ($x=x_0=$constant) be described by

$$X=X_0-\phi(T),\tag{10}$$

with $X=X_0$, $dX/dT=0$, at $T=0$, so that $\phi(0)=0$, $\dot{\phi}(0)=0$. We assume also that $\phi(T)$ is independent of X_0. Mathematically we have

$$x=F[X+\phi(T)]\quad\text{and}\quad t=G(X,T),\tag{11}$$

with ϕ, F, G unknown functions, $F(0)=0$, $G(X,0)=0$.

From Eq. (11) it follows that

$$\frac{dx}{dt}=\frac{F'[X+\phi(T)][(dX/dT)+(d\phi/dT)]}{(\partial G/\partial X)(dX/dT)+(\partial G/\partial T)}.\tag{12}$$

The postulate that $dX/dT=\pm c$ implies that $dx/dt=\pm c$ (coordinate speed of light is an invariant) yields

$$\begin{aligned}c^2(\partial G/\partial X)+c(\partial G/\partial T)&=[c+(d\phi/dT)]F'[X+\phi(T)]\\ \text{and}\qquad&\\ c^2(\partial G/\partial X)-c(\partial G/\partial T)&=[-c+(d\phi/dT)]F'[X+\phi(T)],\end{aligned}\tag{13}$$

so that

$$\begin{aligned}\partial G/\partial X&=(1/c^2)(d\phi/dT)F'[X+\phi(T)]\\ \text{and}\qquad&\\ \partial G/\partial T&=F'[X+\phi(T)].\end{aligned}\tag{14}$$

The integrability condition, $\partial^2G/\partial X\partial T=\partial^2G/\partial T\partial X$, yields

$$\frac{F''[X+\phi(T)]}{F'[X+\phi(T)]}=\frac{(1/c^2)(d^2\phi/dT^2)}{1-(1/c^2)(d\phi/dT)^2}=\frac{g}{c^2}=\text{const.},\tag{15}$$

with g a constant in the dimensions of acceleration.

Integrating Eq. (15) yields

$$\begin{aligned}\phi(T)&=(c^2/g)\ln\cosh(gT/c)\\ \text{and}\qquad&\\ F[X+\phi(T)]&=A+B\exp\left\{\frac{g}{c^2}[X+\phi(T)]\right\}.\end{aligned}\tag{16}$$

From $F(0)=0$ it follows that $A+B=0$, with $A=-c^2/g$ in order that

$$\lim_{g\to 0}F[X+\phi(T)]=X.$$

Thus

$$\begin{aligned}x&=c^2/g[e^{gX/c^2}\cosh(gT/c)-1],\\ \text{and}\qquad&\\ t&=(c/g)e^{gX/c^2}\sinh(gT/c),\end{aligned}\tag{17}$$

by making use of Eqs. (14) and (16).

One notes that

$$\begin{aligned}&\left.\begin{aligned}x&\approx X+\tfrac{1}{2}gT^2\\ t&\approx T\end{aligned}\right\}\quad\text{for}\quad gT/c\ll 1,\quad gX/c^2\ll 1\\ &\lim_{T\to\infty}(dx/dt)_{X=\text{const.}}=c.\end{aligned}\tag{18}$$

From Eq. (17) it follows that

$$\begin{aligned} ds^2 &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= e^{2gX/c^2}[c^2dT^2 - dX^2] - dY^2 - dZ^2 \\ &= g_{\alpha\beta}(X)dX^\alpha dX^\beta, \end{aligned} \tag{19}$$

with $y = Y$, $z = Z$. At the origin of the accelerating frame of reference, proper time is given by $ds/c = dT$ and, in general, it is given by $ds/c = e^{gX/c^2}dT$, which shows that the rate of a clock is a function of the position of the clock.

It is interesting to note that the elements of the metric tensor $g_{\alpha\beta}(X)$ of Eq. (19) satisfy Einstein's field equations $R_{ij} = 0$, with ds^2 the line element due to a uniform gravitational field in the negative X direction. Our $Oxyz$ frame of reference would represent a freely falling coordinate system (inertial).

IV. A TWIN PROBLEM

In Sec. III, O and O' were initially at rest relative to each other. Let $O''\xi\eta\zeta$ be an inertial system coincident with O and O' at $t = T = \tau = 0$, but moving with speed V along the x axis relative to O. We consider O' and O'' as our twins. From the point of view of O'' it appears that O' has been thrown off initially with a speed $-V$; O' becomes coincident with O'' at a later time due to the acceleration of O'. We calculate now the times T_0 and τ_0 when O' and O'' meet for the second time.

From Eq. (17) and the Lorentz–Einstein transformations we obtain

$$\xi = \frac{c^2}{g}(1 - V^2/c^2)^{-\frac{1}{2}} \times \left\{ e^{gX/c^2}\left[\cosh(gT/c) - \frac{V}{c}\sinh(gT/c)\right] - 1 \right\},$$

and

$$\tau = \frac{c}{g}(1 - V^2/c^2)^{-\frac{1}{2}} \times \left\{ e^{gX/c^2}\left[\sinh(gT/c) - \frac{V}{c}\cosh(gT/c)\right] + \frac{V}{c} \right\}. \tag{20}$$

O' and O'' meet again when $\xi = X = 0$, $T = T_0$, $\tau = \tau_0$, so that

$$\begin{aligned} 1 &= \cosh(gT_0/c) - \frac{V}{c}\sinh(gT_0/c), \\ &\text{and} \\ \tau_0 &= \frac{c}{g}(1 - V^2/c^2)^{-\frac{1}{2}} \times \left[\sinh(gT_0/c) - \frac{V}{c}\cosh(gT_0/c) + \frac{V}{c} \right], \end{aligned} \tag{21}$$

from which it follows that

$$\begin{aligned} T_0 &= \frac{c}{g}\ln\frac{1 + V/c}{1 - V/c} \\ &= \frac{2V}{g}\left[1 + \frac{1}{3}\left(\frac{V}{c}\right)^2 + \frac{1}{5}\left(\frac{V}{c}\right)^4 + \cdots \right], \\ &\text{and} \\ \tau_0 &= \frac{2V}{g}(1 - V^2/c^2)^{-\frac{1}{2}} \\ &= \frac{2V}{g}\left[1 + \frac{1}{2}\left(\frac{V}{c}\right)^2 + \frac{3}{8}\left(\frac{V}{c}\right)^4 + \cdots \right]. \end{aligned} \tag{22}$$

Since $T_0 < \tau_0$, it follows that the accelerating twin is younger than the inertial twin when they meet for the second time.

In conclusion, let us return to the classical problem in which twin B accelerates away from twin A, then coasts, decelerates, and eventually returns to A. It is true that as B coasts away from A that A will age less rapidly than B from B's point of view. However, as B decelerates at a distance X from A we note from Eq. (19), replacing g by $-g$, that B's clock will run at a much slower rate than A's clock (by a factor e^{-gX/c^2}). It is precisely during this deceleration period that A becomes older than B, as has been previously noted by most relativists.

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